PMM U.S.S.R.,Vol.44,pp.786-793
0021-8928/81/6 0786 $7.50/0

UDC 539.3

ON THE CONSTRUCTION AND INVESTIGATION OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS OF THERMOVISCOELASTICITY*

S. E. UMANSKII

The uncoupled mixed boundary value problem of thermoviscoelasticity is considered in a quasistatic formulation. The temperature distribution is assumed nonstationary and inhomogeneous. The influence of the temperature on the viscoelastic properties of the material is taken into account by the introduction of a reduced time. The equations of state of the material are written in differential form as a system of kinetic equations in some tensor-type strain parameters. The system mentioned is equivalent to a Volterra integral equation with kernel in the form of a sum of exponents. The differential approach used is apparently more convenient for numerical realization /1/ (especially in nonuniform problems) and results in a substantially different mathematical formulation as compared with that based on the integral form of writing the equations of state investigated in /2,3/. Precisely for going over to the boundary value problem are the kinetic differential equations converted into an operator differential equation in Hilbert space. The existence, uniqueness, and stability of the solution of the problem formulated are established, and conditions for the convergence of the Galerkin approximations and the stability of the difference approximations in time are formulated.

1. Formulation of the Problem.

Let us consider an uncoupled mixed quasi-static boundary value problem of thermoviscoelasticity for a body subjected to nonstationary inhomogeneous heating (cooling) and mechanical loading. We will consider the material linearly viscoelastic, and shall assume that the principle of temperature- and structure-time correspondence is satisfied, i.e. the influence of temperature and structure changes on the viscosity is taken into account by replacement of the real time t by the reduced time ξ defined by the relationship

$$d\xi = g_T dt \tag{1.1}$$

Here g_T is some functional of the temperature history of the material. The governing equations for such material have the form /2/

$$\sigma(\xi) = \int_0^{\xi} \mathbf{c}(\xi - \xi') \frac{\partial[\varepsilon(\xi') - \beta(\xi')]}{\partial\xi'} d\xi' \tag{1.2}$$

Here σ, ε and β are six-dimensional vectors corresponding to the symmetric stress, strain, and unconstrained (thermal and structural) strains, $\mathbf{c}(t)$ is the matrix of the relaxation moduli which has the following form in the case of an isotropic material

$$\mathbf{c}(t) = k(t)\,\mathbf{I}_1 + 2G(t)\,\mathbf{I}_2, \quad I_1 = \frac{1}{3}\left\| \begin{array}{c|c} U^1 & 0 \\ \hline 0 & 0 \end{array} \right\|, \quad I_2 = \frac{1}{3}\left\| \begin{array}{c|c} U^2 & 0 \\ \hline 0 & U^3 \end{array} \right\|$$

$$U^k = \| U_{ij}{}^k \|, \; k = 1, 2, 3, \; i, j = 1, 2, 3$$

$$U_{ij}{}^1 = 1; \; U_{ii}{}^2 = 2, \; U_{ij}{}^2 = -1, \; i \neq j, \quad U_{ii}{}^3 = 3, \; U_{ij}{}^3 = 0, \; i \neq j$$

Here $k(t)$ and $G(t)$ are, respectively, the volume and shear strain moduli I_1, I_2 are 6×6 matrices permitting the separation of the hydrostatic and deviator parts of an arbitrary stress deformation vector.

In a number of cases of practical importance, the relaxation kernels are represented in the form of a sum of exponentials with negative exponents. In this case the equations of state of the material (1.2) are equivalent to the following relationships

$$\sigma = \mathbf{D}(\varepsilon - \beta) - \sum_{m=1}^{M} \mathbf{D}_m \varepsilon_m{}^c \tag{1.3}$$

$$(\varepsilon_m{}^c)^{\cdot} = g_T \mathbf{R}_m (\varepsilon - \varepsilon_m{}^c - \beta), \quad m = 1, \ldots, M \tag{1.4}$$

Here D is a matrix of the elastic moduli of the material, $\varepsilon_m{}^c$ are strain parameters (variable state), $\mathbf{D}_m, \mathbf{R}_m$ are matrices whose components are expressed in terms of coefficients of an exponential expansion of the kernel of (1.2). In the case of an isotropic material

*Prikl.Matem.Mekhan.44,No.6,1100-1110,1980

$$\mathbf{D}_m = k_m\mathbf{I}_1 + 2G_m\mathbf{I}_2, \quad \mathbf{R}_m = \rho_m^{-1}\mathbf{I}_1 + r_m^{-1}\mathbf{I}_2 \tag{1.5}$$

$$\mathbf{D} = k_0\mathbf{I}_1 + 2\,G_0\mathbf{I}_2, \quad k\,(0) = k_0, \quad G\,(0) = G_0, \quad \mathbf{D}_\infty = k_\infty\mathbf{I}_1 + 2G_\infty\mathbf{I}_2$$

Here $k_m, G_m, \rho_m, r_m, k_\infty, G_\infty$ are coefficients determined from the condiction of best approximation of the relaxation curves by analytic dependences

$$G\,(t) = G_\infty + \sum_{m=1}^{M} G_m \exp\left(-\frac{t}{r_m}\right), \quad k\,(t) = k_\infty + \sum_{m=1}^{M} k_m \exp\left(-\frac{t}{\rho_m}\right) \tag{1.6}$$

The equations of state written in the differential form (1.3), (1.4) result in a substantially different, and apparently, more convenient formulation of the boundary value problem of numerical realization (see /1/) as compared with that investigated in /2,3,4/, based on the integral representation of the governing equations. Nevertheless, as mentioned above, the initial relationships (1.3)—(1.5) and (1.2), (1.6) are equivalent. Indeed, we introduce into the consideration the quantity $\sigma_m = \mathbf{D}_m(\varepsilon - \beta - \varepsilon_m^c)$ and we convert the system (1.3), (1.4) to the form

$$\sigma = \sum_m \sigma_m - \Big(\mathbf{D} - \sum_m \mathbf{D}_m\Big)(\varepsilon - \beta), \quad \dot{\sigma}_m + \mathbf{R}_m\sigma_m = \mathbf{D}_m(\varepsilon^{\cdot} - \beta^{\cdot})$$

from which (1.2) then follows.

The form of the matrices $\mathbf{D}, \mathbf{c}\,(t), \mathbf{D}_m, \mathbf{R}_m$ is determined by the nature of the anisotropy in the case of anisotropic material. Essential for the sequel is just the requirement that the matrix $\mathbf{D}$ be symmetric, positive definite, bounded, and measurable in the domain Ω under consideration, $\mathbf{D}_m$ and $\mathbf{R}_m$ should be symmetric, nonnegative definite, and piecewise constant in Ω, the matrix

$$\mathbf{D}_\infty = \mathbf{D} - \sum_{m=1}^{M} \mathbf{D}_m = \lim_{t\to\infty} \mathbf{c}\,(t) \tag{1.7}$$

be positive definite, and that there exist a symmetric, nonnegative definite matrix $\mathbf{H}_m$ such that

$$\mathbf{H}_m\mathbf{R}_m = \mathbf{R}_m\mathbf{H}_m = \mathbf{D}_m \tag{1.8}$$

$$0 < \gamma = \sup_\Omega \max_m \sup_{\varepsilon_m^c \in R^6} \frac{\{\mathbf{H}_m\varepsilon_m^c,\ \varepsilon_m^c\}}{\{\mathbf{D}\varepsilon_m^c,\ \varepsilon_m^c\}} \tag{1.9}$$

Here $\{\ ,\ \}$ is the scalar product of six-dimensional vectors defined as the convolution of their corresponding symmetric tensors of the second rank.

The inequalities

$$0 < \tau_2 = \sup_\Omega \max_m \sup_{\varepsilon_m^c \in R^6} \frac{\{\mathbf{R}_m\varepsilon_m^c,\ \varepsilon_m^c\}}{\{\varepsilon_m^c,\ \varepsilon_m^c\}} \tag{1.10}$$

$$1 \leqslant \alpha = \max\,[1,\ \sum_m \|\mathbf{Q}_m\|_{L_\infty(\Omega)}], \quad \mathbf{Q}_m = \sup_{\varepsilon\in R^6} \frac{\{\mathbf{D}_m\varepsilon,\ \varepsilon\}}{\{\mathbf{D}_\infty\varepsilon,\ \varepsilon\}} \in L_\infty(\Omega) \tag{1.11}$$

also follow from the conditions formulated above.

With respect to the reduction function g_T we assume that it is measurable and bounded at each time t and there is compliance with the inequality

$$0 < g_0 \leqslant g_1\,(t) \leqslant \operatorname{vrai}_\Omega \min g_T\,(t) \leqslant \operatorname{vrai}_\Omega \max g_T\,(t) \leqslant g_2\,(t) \leqslant g_3 < \infty \tag{1.12}$$

Here g_0, g_3 are constants, $g_1(t), g_2(t)$ are continuous functions of the time, $\operatorname{vrai}_\Omega \min g_T$ and $\operatorname{vrai}_\Omega \max g_T$ are true, i.e., the minimum and maximum of the function $g_T \in L_\infty(\Omega)$ /5/ used without taking into account isolated points and sets of measure zero.

Let the domain under consideration $\Omega \in R^3$ be bounded and have the regular boundary $\partial\Omega = \partial_1\Omega \vee \partial_2\Omega$, where homogeneous boundary conditions in displacements are given on the section $\partial_1\Omega$, surface loads φ are applied on the section $\partial_2\Omega$, and in addition, mass forces $\mathbf{f}$ act on the body. We consider that

$$\varphi \in C^\circ\,[S, (H^\circ(\partial_2\Omega))^3], \quad \mathbf{f} \in C^\circ\,[S, (H^\circ(\Omega))^3]$$

Here $S = [0, T]$ or $[0, \infty]$ is the time interval, $C^k\,[S, \Sigma]$ is the space of functions from S continuously differentiable k times with respect to the time in an arbitrary Banach space Σ, $(H^l(\Omega))^n$ is the Sobolev space of order l over the vector functions from $(\Omega \to R^n)$, where $(X \to Y)$ denotes the set of all possible mappings of the set X into the set Y.

We consider the strains and displacements related by means of the Cauchy relationships

$$\varepsilon_{ij} = {}^1/_2\,(\partial u_j / \partial x_i + \partial u_i / \partial x_j)$$

which can be written in the operator form

$$\varepsilon = \mathbf{B}\mathbf{u} \tag{1.13}$$

Here $\mathbf{u}$ is the displacement vector, and $\mathbf{B}$ is a bounded linear operator acting from the space of displacement fields

$$V = \{\mathbf{v} \mid \mathbf{v} \in (H^1(\Omega))^3,\ \mathbf{v}|_{\partial_1\Omega} = 0\}$$

into the Hilbert space Y whose elements are the strain (stress) vectors, considered as vector functions in Ω with the scalar product

$$(\varepsilon,\ \varkappa)_Y = \int_\Omega \{\varepsilon,\ \varkappa\}\, d\Omega = \int_\Omega \varepsilon_{ij}\varkappa_{ij}\, d\Omega, \quad \forall \varepsilon,\ \varkappa \in Y \tag{1.14}$$

It is assumed that the set V is also allotted a scalar product

$$(\mathbf{w},\ \mathbf{v})_V = (\mathbf{B}\mathbf{w},\ \mathbf{B}\mathbf{v})_Y = \int_\Omega \{\mathbf{B}\mathbf{w},\ \mathbf{B}\mathbf{v}\}\, d\Omega, \quad \forall \mathbf{w},\ \mathbf{v} \in V \tag{1.15}$$

communicating the structure of a Hilbert space to it.

2. Existence, uniqueness, and properties of the generalized solution of the boundary value problem of thermoviscoelasticity. We write the Lagrange variational equation in the form

$$(\sigma,\ \mathbf{B}\mathbf{v})_Y = \int_{\partial_2\Omega} \varphi_j v_j\, ds + \int_\Omega f_j v_j\, d\Omega, \quad \forall \mathbf{v} \in V \tag{2.1}$$

Let the symbol Y^* denote the space conjugate to Y, and let us identify Y and Y^* in conformity with the Riesz theorem. We define the function $\mu \in C[S, Y]$ in such a manner that $\mu(t)$ for any fixed t is a continuation of the functional

$$L(\mathbf{B}\mathbf{v}) = \int_{\partial_2\Omega} \varphi_j(t)\, v_j\, ds$$

given in a set of values of the operator B in the space Y, into the whole space $Y^* = Y$. Then we write the integral identity (1.1) as the operator equation

$$\mathbf{B}^*\sigma = \mathbf{B}^*\mu + \mathbf{f} \tag{2.2}$$

Here $\mathbf{B}^*: Y^* \to V^*$ is the operator conjugate to $\mathbf{B}$, and V^* is the space conjugate to V. Substituting (1.3) into (2.2), we obtain the equilibrium equation in displacements

$$\mathbf{K}\mathbf{u} = \mathbf{f} + \mathbf{B}^*\rho, \quad \rho = \mu + \mathbf{D}\beta + \sum_{m=1}^{M} \mathbf{D}_m \varepsilon_m{}^c \tag{2.3}$$

Here $\mathbf{K} = \mathbf{B}^*\mathbf{D}\mathbf{B}$ is an elliptical operator from V into V^* corresponding to the homogeneous boundary value problem of the theory of elasticity. The operator $\mathbf{K}$ is self-adjoint because of the reflexivity of the space V. Moreover, as follows from the results in /4,5/, it possesses a bounded inverse $\mathbf{K}^{-1}$, and (2.3) is therefore solvable uniquely (for given $\varepsilon_m{}^c$) at any fixed time $t \in S$.

Finding u from (2.3), and then using (1.13) and (1.4), we have

$$(\varepsilon_m{}^c)' + g_T\mathbf{R}_m\left(\varepsilon_m{}^c - \theta \sum_{k=1}^{M} \mathbf{D}_k\varepsilon_k{}^c\right) = g_T\mathbf{R}_m\,(\mathbf{B}\mathbf{K}^{-1}(f + \mathbf{B}^*\mathbf{D}\beta + \mathbf{B}^*\mu) - \beta),\ m = 1, \ldots, M,\ \beta \in C[S, Y] \tag{2.4}$$

Here $\theta = \mathbf{B}\mathbf{K}^{-1}\mathbf{B}^*$ is a self-adjoint, nonnegative definite operator in the space Y.

Let us define a *finite set of scalar products of the form*

$$(\eta,\ \varkappa)_m = \int_\Omega \{\mathbf{H}_m\eta,\ \varkappa\}\, d\Omega = (\mathbf{H}_m\eta,\ \varkappa)_Y, \quad m = 1, \ldots, M \tag{2.5}$$

in the set Y.

We call Y_m the set Y with the scalar product (2.5), and we denote the identity mapping $Y \to Y_m$ by i_m and examine the direct sum X of all Y_m $(m = 1, \ldots, M)$. The set $X = Y_1 \oplus Y_2 \oplus \ldots \oplus Y_M$ consists of all possible ordered sets of vector-functions from Y_m identifiable with the corresponding state parameters.

Let $\zeta = (\varepsilon_1{}^c, \ldots, \varepsilon_m{}^c, \ldots, \varepsilon^c{}_M)$ and $\eta_m{}^c$ be arbitrary elements of the spaces X and Y_m. We define the projection $\mathbf{Q}_m: X \to Y_m$ and imbedding $\mathbf{L}_m: Y_m \to X$ operators in such a way that

$$\mathbf{Q}_m\zeta = \varepsilon_m{}^c \in Y_m, \quad \mathbf{L}_m\eta_m{}^c = (0, \ldots, \eta_m{}^c,\ 0, \ldots, 0) \in X$$

The set X becomes a Hilbert space if a scalar product is given therein by using the relationship

$$(\zeta, \chi) = \sum_{m=1}^{M} (\mathbf{Q}_m\zeta, \mathbf{Q}_m\chi) = \sum_{m=1}^{M} (\mathbf{H}_m\mathbf{Q}_m\zeta, \mathbf{Q}_m\chi)_Y, \quad \forall\zeta, \chi \in X \tag{2.6}$$

Using the space X and the system of equations (2.4), we reduce the problem of thermoviscoelasticity under consideration to a Cauchy problem for the following operator differential equation

$$\zeta^{\cdot} + g_T\Pi\zeta = \psi, \ \zeta \in C^1[S, X], \ \zeta(0) = \zeta_0 \in X \tag{2.7}$$

Here the operator $\Pi : X \to X$ and the function $\psi \in C[S, X]$ are represented in conformity with (2.4) in the form

$$\Pi = \sum_{m=1}^{M} \mathbf{L}_m \left(\mathbf{R}_m - R_m\theta \sum_{k=j}^{M} \mathbf{D}_k\mathbf{Q}_k\right), \quad \psi = \sum_{m=1}^{M} g_T\mathbf{L}_m\mathbf{R}_m(\mathbf{BK}^{-1}(\mathbf{f} + \mathbf{B}^*\mathbf{D}\beta + \mathbf{B}^*\mu) - \beta) \tag{2.8}$$

Lemma 2.1. The linear operator $\Pi : X \to X$ is self-adjoint, coercive, and bounded, where the inequality

$$\tau_1 \| \zeta \|^2 \leqslant (\Pi\zeta, \zeta) \leqslant \tau_2 \| \zeta \|^2 \tag{2.9}$$

is correct for $\tau_1 = 1/(\alpha\gamma), \tau_2$ positive constants defined by (1.9)–(1.11).

Proof. Taking account of (2.6), (1.8) and the symmetry of the matrix $\mathbf{D}_m$, the expression $(\Pi\zeta, \chi)$ has the form

$$(\Pi\zeta, \chi) = \sum_{m=1}^{M} (\mathbf{D}_m\mathbf{Q}_m\zeta, \mathbf{Q}_m\chi)_Y - \Big(\sum_{m=1}^{M} \mathbf{D}_m\mathbf{Q}_m\zeta, \theta\sum_{m=1}^{M} \mathbf{D}_m\mathbf{Q}_m\chi\Big)_Y \tag{2.10}$$

Hence, because of the self-adjointness of θ, the self-adjointness of the operator Π follows.

To prove the inequality (2.9), we consider a hypothetical body identical to that given in shape and viscoelastic properties, but free of external loads including the thermal. Let a certain continuously time-varying distribution of the state parameters $\{\varepsilon_m^c\}_{m=1}^M$ to which the function

$$\zeta(t) = \sum_{m=1}^{M} \mathbf{L}_m\varepsilon_m^c(t) \in C[S, X]$$

corresponds, originate in the body mentioned under the influence of chemical transformations.

In this case

$$\varepsilon = \theta\sum_{k=1}^{M} \mathbf{D}_k\varepsilon_k^c, \quad (\Pi\zeta, \zeta) = \sum_{m=1}^{M} (\mathbf{D}_m(\varepsilon_m^c - \varepsilon), \varepsilon_m^c)_Y, \tag{2.11}$$

The rate of change of the additional energy E' for the body under consideration at any time is

$$E'^{\cdot} = (\sigma^{\cdot}, \varepsilon)_Y = \langle\mathbf{B}^*\sigma^{\cdot}, \mathbf{u}\rangle = \langle 0, \mathbf{u}\rangle = 0$$

On the other hand, taking account of (1.3) and (1.7), we have

$$E'^{\cdot} = \Big(\mathbf{D}\varepsilon^{\cdot} - \sum_{m=1}^{M} \mathbf{D}_m\varepsilon_m^{c\cdot}, \varepsilon\Big)_Y = \frac{1}{2}\frac{d(\mathbf{D}_\alpha\varepsilon, \varepsilon)}{dt} + \frac{1}{2}\frac{d}{dt}\Big[\sum_{m=1}^{M} (\mathbf{D}_m(\varepsilon - \varepsilon_m^c), \varepsilon - \varepsilon_m^c)_Y\Big] + \sum_{m=1}^{M} (\mathbf{D}_m(\varepsilon^{\cdot} - \varepsilon_m^{c\cdot}), \varepsilon_m^c)_Y$$

Because of (2.10) and (2.11), this last component is $(\Pi\zeta^{\cdot}, \zeta) = -\frac{1}{2}(\Pi\zeta, \zeta)$, from which by taking into account that $E'^{\cdot} = 0$, we obtain

$$(\Pi\zeta, \zeta) = \sum_m (\mathbf{D}_m(\varepsilon - \varepsilon_m^c), \varepsilon - \varepsilon_m^c)_Y + (\mathbf{D}_\alpha\varepsilon, \varepsilon)_Y \geqslant 0$$

Using (1.11), the triangle inequality, (1.9) and (2.6), we have

$$(\Pi\zeta, \zeta) \geqslant \alpha^{-1}\Big[\sum_{m=1}^{M} (\mathbf{D}_m(\varepsilon - \varepsilon_m^c), \varepsilon - \varepsilon_m^c)_Y + \sum_{m=1}^{M} (\mathbf{D}_m\varepsilon, \varepsilon)_Y\Big] \geqslant \alpha^{-1}\sum_{m=1}^{M} (\mathbf{D}_m\varepsilon_m^c, \varepsilon_m^c)_Y \geqslant (\alpha\gamma)^{-1}\sum_{m=1}^{M} (\mathbf{H}_m\varepsilon_m^c, \varepsilon_m^c)_Y = (\alpha\gamma)^{-1}\|\zeta\|^2$$

Taking into account the nonnegative-definiteness of the operator θ and the expressions (1.8)–(1.10), we obtain from (2.10) for $\chi = \zeta$

$$(\Pi\zeta, \zeta) = \sum_m (\mathbf{D}_m\varepsilon_m^c, \varepsilon_m^c) - \Big(\sum_m \mathbf{D}_m\varepsilon_m^c, \theta\sum_m \mathbf{D}_m\varepsilon_m^c\Big)_Y \leqslant \sum_m (\mathbf{D}_m\varepsilon_m^c, \varepsilon_m^c)_Y \leqslant \tau_2\|\zeta\|^2$$

The lemma is proved.

The unique solvability of the Cauchy problem for (2.7) and the boundedness of the solution on the half-axis $[0, \infty)$ follow from Lemma 2.1, the inequality (1.12), and the general results for operator differential equations (see /6/, for instance).

Theorem 2.1. Let $\zeta(t)$ be the solution of (2.7) that satisfies the condition $\zeta(0) = \zeta_0$. Then the estimate

$$\|\zeta(t)\| \leqslant \|\zeta_0\| \exp\Big(-\tau_1 \int_0^t g_1(s)\,ds\Big) + \int_0^t \|\psi(p)\| \exp\Big(-\tau_1 \int_0^p g_1(s)\,ds\Big)\,dp \tag{2.12}$$

is valid for the norm $\|\zeta\|$.

Proof. We multiply (2.7) scalarly by ζ. Taking into account that $(\zeta^{\cdot}, \zeta) = {}^1/_2 \|\zeta\|^{\cdot}$ we have

$$\|\zeta\|\, d\|\zeta\| / dt = (\psi, \zeta) - (g_T \Pi\zeta, \zeta)$$

Using the Schwarz inequality as well as (1.12) and (2.8), we obtain

$$d\|\zeta\| / dt \leqslant \|\psi\| - g_1(t)\tau_1 \|\zeta\| \tag{2.13}$$

The estimate (2.12) follows from the inequality (2.13) and the Gronwall lemma (see /7/, for example).

3. Convergence of the Galerkin approximations. Let us determine the one-parameter families $\{W_h\}$ and. $\{Z_h\}$ of approximate finite-dimensional spaces whose dimensionality will grow without limit under the condition that some common parameter h for these families tends to zero. Here $\{W_h\}$ is a sequence of finite Hilbert spaces, each of which is isomorphic to a certain closed subspace $V_h \in V$, where $\mathbf{P}_h : W_h \to V$ is an operator setting up the isomorphism mentioned and thereby being the operator imbedding W_h into V, $\{Z_h\}$ is a sequence of finite Hilbert spaces such that for any Z_h there exists an operator $\mathrm{S}_h : Z_h \to Y$ setting up an isomorphism between Z_h and the space $Y_h \subset Y$; $\mathbf{P}_h^* : V^* \to W_h^*$ and $\mathrm{S}_h : Y = Y^* \to Z_h = Z_h$ are, respectively, operators adjoint to $\mathbf{P}_h$ and S_h and defined by the relationships (3.6)

$$(\mathrm{S}_h^* \mathbf{e}, \varkappa_h)_{Z_h} = (\mathbf{e}, \mathrm{S}_h \varkappa_h)_Y, \ \forall \mathbf{e} \in Y^* = Y, \ \varkappa_h \in Z_h, \ \langle \mathbf{P}_h^* f, \mathbf{w}_h \rangle_h = \langle \mathbf{f}, \mathbf{P}_h \mathbf{w}_h \rangle, \ \forall \mathbf{f} \in V^*, \ \mathbf{w}_h \in W_h \tag{3.1}$$

Here $\langle \mathbf{f}, \mathbf{v}\rangle$ denotes the action of a functional $\mathbf{f} \in V^*$ on an element of the space V.

The scalar products in the spaces Z_h and W_h are given as follows:

$$(\mathbf{z}_1, \mathbf{z}_2)_{Z_h} = (\mathrm{S}_h \mathbf{z}_1, \mathrm{S}_h \mathbf{z}_2)_Y, \quad \forall \mathbf{z}_1 \mathbf{z}_2 \in Z_h, \ (\mathbf{w}_1, \mathbf{w}_2)_{W_h} = (\mathbf{P}_h \mathbf{w}_1, \mathbf{P}_h \mathbf{w}_2)_V, \quad \forall \mathbf{w}_1, \mathbf{w}_2 \in W_h \tag{3.2}$$

It is assumed that

$$\mathbf{B}\mathbf{P}_h \mathbf{w}_h \in Y_h, \quad \forall \mathbf{w}_h \in W_h$$

Then the operator $\mathbf{B}_h = \mathrm{S}_h^* \mathbf{B} \mathbf{P}_h \in (W_h \to Z_h)$ is correctly dermined.

Let us set up a correspondence between the notation introduced and the terminology taken in the literature by the method of finite elements /8/, which is a variation of the Galerkin method. In this case h is the maximum among the diameters of the elements on which the initial domain is separated, W_h is the set of nodal displacements vectors satisfying the kinematic boundary conditions, Z_h is the set of strain values of characteristic points of the element (for instance, at the centers of gravity or other weighting points), the operator P_h is defined by using a function of the shape of the element, and the operator $\mathbf{B}_h$ within each element corresponds to the matrix transforming the nodal displacements into strains.

Let us note that because of the definition (3.2) of the scalar products, the operators

$$\mathbf{K}_h = \mathbf{P}_h^* \mathbf{K} \mathbf{P}_h \in (W_h \to W_h^*), \quad \theta_h = \mathbf{B}_h \mathbf{K}_h^{-1} \mathbf{B}_h^* \in (Z_h \to Z_h)$$

conserve the fundamental properties of the operators $\mathbf{K}$ and θ, respectively. In particular, they are self-adjoint and bounded, the operator θ_h is nonnegative definite, and $\mathbf{K}_h$ is coercive, and therefore, possesses a bounded inverse $\mathbf{K}_h^{-1}$ so that for any $\mathbf{f} \in V^*$, $\rho \in Y$ the equation

$$\mathbf{K}_h \mathbf{u}_h = \mathbf{P}_h \mathbf{f} + \mathbf{B}_h^* \mathrm{S}_h \rho, \quad \rho = \mu + \mathbf{D}\beta + \sum_{m=1}^{M} \mathbf{D}_m \varepsilon_m^c \tag{3.3}$$

approximating the problem (2.3) is uniquely solvable.

We call the corresponding approximation convergent if

$$\lim_{h \to 0} \|\mathrm{S}_h \mathbf{B}_h \mathbf{K}_h^{-1} (\mathbf{P}_h^* \mathbf{f} + \mathbf{B}_h^* \mathrm{S}_h^* \rho) - \mathbf{B}\mathbf{K}^{-1} (\mathbf{f} + \mathbf{B}^* \rho)\|_Y = 0, \quad \forall \rho \in Y, \quad \mathbf{f} \in V^* \tag{3.4}$$

$$\lim_{h \to 0} \|\mathrm{S}_h \mathrm{S}_h^* - \mathbf{I}_Y\|_Y = 0 \tag{3.5}$$

Compliance with the relationships (3.4) and (3.5) is assured by the selection of the families $\{W_h\}$ and $\{Z_h\}$. In particular, a sufficient condition for convergence when using the method of finite elements is the requirement of homogeneity of the basis /9/, which reduces in the case of triangular elements to the fact that the minimum angle in the triangles will remain greater than a certain fixed constant as the mesh is made finer.

Let us construct an approximate finite space E_h for the space X. We consider an ordered set of spaces $\{Z_{mh}\}_{m=1}^{M}$, each of which consists of the same elements as the space Z_h but is alloted a scalar product of the form $(\eta_h, \varkappa_h)_{mh} = (\mathbf{H}_m\eta_h, \varkappa_h)_{Z_h}$. Let us define E_h as their direct sum $E_h = Z_{1h} \oplus Z_{2h} \oplus \ldots \oplus Z_{Mh}$. The canonical projector $\theta_{mh} : E_h \to Z_{mh}$ and the canonical imbedding operator $\mathbf{L}_{mh}: Z_{mh} \to E_h$ are identical to the operators $\mathbf{L}_m$ and θ_m introduced in Sect. 2, and the scalar product E_h has the form

$$(\zeta_h, \chi_h)_{E_h} = \sum_{m=1}^{M} (\mathbf{H}_m \mathbf{Q}_{mh}\zeta_h, \mathbf{Q}_{mh}\chi_h)_{Z_h}, \quad \forall \zeta_h \chi_h \in E_h$$

Let $\mathbf{i}_{mh}$ be an operator assigning the identity mapping $Z_h \to Z_{mh}$. Then the operator

$$\mathbf{J}_h = \sum_{m=1}^{M} \mathbf{L}_m \mathbf{i}_m \mathbf{S}_h \mathbf{i}_{mh}^{-1} \mathbf{Q}_{mh} \in (E_h \to X) \tag{3.7}$$

defines the isomorphism between E_h and some closed subspace $X_h \in X$, where $\| \mathbf{J}_h\zeta_h \|_X = \| \zeta \|_{E_h}$. Let $\mathbf{J}_h^* : X^* = X \to E_h^* = E_h$ be an adjoint operator to $\mathbf{J}_h$ (according to the Riesz theorem, we identify E_h^* with E_h and X^* with X). For the operator $\mathbf{I}_h$ we have $\mathbf{J}_h^*\mathbf{J}_h = \mathbf{I}_{E_h}$ and in conformity with (3.7) and (3.5)

$$\lim_{h \to 0} \| \mathbf{J}_h \mathbf{J}_h^* - \mathbf{I}_X \| = 0 \tag{3.8}$$

Here $\mathbf{I}_X, \mathbf{I}_{E_h}$ are unit operators in the appropriate spaces.

Let us define the operators $\mathbf{\Pi}_h \in (E_h \to E_h)$, $g_{Th} \in C[S, (E_h \to E_h)]$ and the vector function $\psi_h \in C[S, E_h]$ by using the relationships

$$\mathbf{\Pi}_h = \sum_{m=1}^{M} \mathbf{L}_{mh} (\mathbf{R}_m - \mathbf{R}_m \theta_h \sum_{k=1}^{M} \mathbf{D}_k \mathbf{Q}_{kh}), \quad g_{Th} = J_h^* g_T J_h \tag{3.9}$$

$$\psi_h = g_{Th} \sum_{m=1}^{M} \mathbf{L}_{mh} \mathbf{R}_m [\mathbf{B}_h \mathbf{K}_h^{-1} (\mathbf{P}_h \mathbf{f} + \mathbf{B}_h^* \mathbf{S}_h^* (\mathbf{D}\beta + \mu)) - \mathbf{S}_h^* \beta]$$

We consider the Cauchy problem

$$\zeta_h^{\cdot} + g_{Th} \mathbf{\Pi}_h \zeta_h = \psi_h, \ \zeta_h \in C^1 [S, E_h], \ \zeta_h(0) = J_h^* \zeta_0 \tag{3.10}$$

which is the finite-dimensional analog of the problem (2.7).

Because of the definition of the scalar product (3.6), the operators $\mathbf{\Pi}_h$ and g_{Th} conserve the fundamental properties of the operators $\mathbf{\Pi}$ and g_T, in particular, inequalities analogous to (2.9) and (1.12) are satisfied, and therefore the problem (3.10) is uniquely solvable and its solution $\zeta_h(t)$ is subject to an estimate identical to (2.12).

We call the quantity $\mathbf{J}_h\zeta_h$ the Galerkin approximation of the solution of (2.7) and we estimate the magnitude of the error $\delta = \mathbf{J}_h\zeta_h - \zeta$.

Let the operator J_h act on (3.10) from the left, and we subtract (2.7) from the result. Using the expansion $\mathbf{I}_{E_h} = J_h^* J_h$ and the second formula in (3.9), we find

$$\begin{gathered}
\delta^{\cdot} = -(\mathbf{J}_h g_{Th} \mathbf{J}_h^*)(\mathbf{J}_h \mathbf{\Pi}_h \mathbf{J}_h^*)\delta + \mathbf{A}(t), \quad \delta(0) = \delta_0 = \mathbf{J}_h \mathbf{J}_h^* \zeta_0 - \zeta_0 \\
\mathbf{A}(t) = \mathbf{A}_1(t) + \mathbf{A}_2(t) + \mathbf{A}_3(t), \quad \mathbf{A}_1(t) = \mathbf{J}_h \psi_h - \psi \\
\mathbf{A}_2(t) = (\mathbf{I}_X - \mathbf{J}_h \mathbf{J}_h^*) g_T (\mathbf{J}_h \mathbf{\Pi}_h \mathbf{J}_h^*) \zeta, \quad \mathbf{A}_3(t) = g_T (\mathbf{\Pi} - \mathbf{J}_h \mathbf{\Pi}_h \mathbf{J}_h^*) \zeta
\end{gathered} \tag{3.11}$$

Since the operator $(\mathbf{J}_h g_{Th} \mathbf{J}_h^*)(\mathbf{J}_h \mathbf{\Pi}_h \mathbf{J}_h^*) \in C[S(X_h \to X_h)]$ possesses the same properties as the operator $g_T\mathbf{\Pi}$, the estimate

$$\| \delta(t) \| \leqslant \| \delta_0 \| \exp\left(-\tau_1 \int_0^t g_1(s)\, ds\right) + \int_0^t \| \mathbf{A}(p) \| \exp\left(-\tau_1 \int_0^p g_1(s)\, ds\right) dp \tag{3.12}$$

follows from (3.11) in conformity with Theorem 2.1.

Using (3.4), (3.5) and (3.7), we obtain

$$\lim_{h \to 0} \sup_{t \in s} \| A_1(t) \| = 0 \tag{3.13}$$

In conformity with (3.8)

$$\lim_{h \to 0} \| \delta_0 \| = 0, \quad \lim_{h \to 0} \sup_{t \in s} \| A_2(t) \| = 0 \tag{3.14}$$

According to (1.12), $\| \mathbf{A}_3(t) \| \leqslant g_2(t) \| \mathbf{a}(t) \|$, where $\mathbf{a}(t) = \mathbf{\Pi}\zeta - \mathbf{J}_h \mathbf{\Pi}_h \mathbf{J}_h^* \zeta$. Let

$$\mathbf{Q}_m \zeta = \varepsilon_m^c, \quad \sum_m \mathbf{D}_m \mathbf{Q}_m \zeta = \mathbf{q}, \quad \mathbf{Q}_m \mathbf{a}(t) = \mathbf{a}_m$$

Taking into account (2.8), (3.7) and (3.9), we have

$$\mathbf{a}_m(t) = \mathbf{R}_m [(\mathbf{e}_m^c - S_h S_h^* \, \mathbf{e}_m^c) + (S_h \theta_h S_h^* \mathbf{q} - \theta \mathbf{q})]$$

Hence, taking account of (3.5), the definitions of the operators θ and θ_h of (3.4), the triangle inequality, as well as the boundedness of $\|\zeta(t)\|$ and $g_2(t)$ there follows

$$\lim_{h\to 0} \sup_{t\in s} \| A_3(t) \| = 0 \tag{3.15}$$

There results from (3.13)—(3.15) that $\lim_{h\to 0} \sup_{t\in s} \| \delta(t) \| = 0.$

The following assertion is therefore proved.

Theorem 3.1. Under the assumptions (1.8)—(1.12), (3.4) and (3.5), the approximate solution of the thermoviscoelasticity problem obtained by the Galerkin method converges uniformly in time to the exact solution.

4. Difference approximation of the derivatives with respect to time. As follows from the results in Sect.3 for the definition of the approximate Galerkin solution of (2.7), it is necessary to solve the Cauchy problem for the system of ordinary differential equations (3.10). Let us consider an approximate (stepwise) method of solving (3.10) by replacing the time-derivatives by finite differences.

Let $\{t_j\}_{j=1}^N$ be some partition of the time interval S under consideration, $\Delta t_j = t_j - t_j$, ζ_h^j is an approximate value of the function $\zeta_h(t_j)$ obtained by a stepwise method

$$\mathbf{g}_{Th}^j = \frac{1}{\Delta t_j} \int_{t_j}^{t_{j+1}} \mathbf{g}_{Th} dt, \; \psi_h{}^j = \frac{1}{\Delta t_j} \int_{t_j}^{t_{j+1}} \psi_h(t)\, dt$$

Then, in conformity with the explicit difference approximation for (3.10), we obtain

$$\zeta_h^{j+1} = (\mathbf{I}_{E_h} - \Delta t_j \mathbf{g}_{Th}^j \, \Pi_h)\zeta_h{}^j + \Delta t_j \psi_h{}^j \tag{4.1}$$

When using the implicit approximation with weights we have

$$(\mathbf{I}_{E_h} + \omega_1 \Delta t_j \mathbf{g}_{Th}^j \Pi_h)\, \zeta_h^{j+1} = \Delta t_j \psi_h{}^j + (\mathbf{I}_{E_h} - \omega_2 \Delta t_j \mathbf{g}_{Th}^j \Pi_h)\, \zeta_h{}^j \tag{4.2}$$

Here ω_1, ω_2 are the weight factors, $\omega_1 + \omega_2 = 1$. The implicit difference scheme (4.2) is absolutely stable for $\omega_1 \geqslant 1/2$, however, since the operator $(I_{E_h} + \omega_1 \Delta t_j \mathbf{g}_{Th}^j \Pi_h)^{-1}$ is impossible to evaluate in explicit form, as a rule, the solution of (4.2) is determined by iterations by the formulas

$$\zeta_h^{j+1} = \lim_{k\to\infty} \zeta_h^{j+1,\,k}, \quad \zeta_h^{j+1,\,k} = -\omega_1 \Delta t_j \mathbf{g}_{Th} \Pi_h \zeta_h^{j+1,\,k-1} + (\mathbf{I}_{E_h} - \omega_2 \Delta t_j \mathbf{g}_{Th} \Pi_h)\, \zeta_h{}^j + \Delta t_j \psi_h{}^j \tag{4.3}$$

From (4.2) and (4.3) we obtain

$$\zeta_h^{j+1} - \zeta_h^{j+1,\,k} = -\omega_1 \Delta t_j \mathbf{g}_{Th}^{j+1} \Pi_h (\zeta_h^{j+1} - \zeta_h^{j+1,\,k-1})$$

Therefore, it is sufficient for convergence of the iteration process (4.3) that $\omega_1 \Delta t_j \| \mathbf{g}_{Th}^j \Pi_h \| < 1$, from which by taking account of the estimates for the norms of $\mathbf{g}_{Th}$ and Π_h resulting from inequalities that are even valid, as mentioned above, for finite-dimensional operators, we have the criterion for time spacing selection

$$\Delta t_j \leqslant \frac{1}{\omega_1 \tau_1 g_{2j}}, \quad g_{2j} = \frac{1}{\Delta t_j} \int_{t_j}^{t_{j+1}} g_2(t)\, dt \tag{4.4}$$

To assure the stability of the explicit scheme (4.1) it is sufficient to require /10/ that the norm of the transition operator $\Gamma_h{}^j = \mathbf{I}_{E_h} - \Delta t_j \mathbf{g}_{Th}^j \Pi_h$ not be greater than one. Estimation of the norm of $\Gamma_h{}^j$ is made difficult by the fact that the operator $\Gamma_h{}^j$ is non-self-adjoint because of the noncommutativity of $\mathbf{g}_{Th}$ and Π_h. The operator $\Gamma_h{}^j$ becomes self-adjoint during renormalization of the space E_h by using the energetic scalar product $(\zeta_h \chi_h)_{\Pi_h} = (\Pi_h \zeta_h, \chi_h)_{E_h}$. Taking into account that for a self-adjoint positive definite operator Π_h there exists a self-adjoint operator $\Pi_h^{1/2}$ such that $\Pi_h^{1/2}\, \Pi_h^{1/2} = \Pi_h$, we have

$$\| \Gamma_h{}^j \|_{\Pi_h} = \sup_{\zeta_h \in E_h} \frac{|(\Pi_h \Gamma_h{}^j \zeta_h, \zeta_h)_{E_h}|}{(\Pi_h \zeta_h, \zeta_h)_{E_h}} = \sup_{\chi_h} \left| 1 - \frac{\Delta t_j (\Pi_h^{1/2} \mathbf{g}_{Th} \Pi_h^{1/2} \chi_h, \chi_h)_{E_h}}{(\chi_h, \chi_h)_{E_h}} \right| = \max\{Q_1, Q_2\} \tag{4.5}$$

$$\chi_h = \Pi_h^{1/2} \zeta_h \in E_h, \; Q_1 = 1 - \Delta t_j \inf_{\chi_h} \frac{(\Pi_h^{1/2} \mathbf{g}_{Th} \Pi_h^{1/2} \chi_h, \chi_h)_{E_h}}{(\chi_h, \chi_h)_{E_h}}, \quad Q_2 = \Delta t_j \sup_{\chi_h} \frac{(\Pi_h^{1/2} \mathbf{g}_{Th} \Pi_h^{1/2} \chi_h, \chi_h)_{E_h}}{(\chi_h, \chi_h)_{E_h}} - 1$$

The quantity $Q_1 \leqslant 1$ because of nonnegativity of the operator g_{Th}. For $Q_2 \leqslant 1$ it is sufficient to require

$$\Delta t_j \leqslant 2 / (\tau_2 g_{2j}) \leqslant 2 / \| \Pi_h^{1/2} g_{Th}^j \Pi_h^{1/2} \| \tag{4.6}$$

Condition (4.6) is the sufficient condition for stability of the explicit scheme (4.1). The scheme (4.1) and (4.2) are first and second order approximations, respectively, which in combination with the stability assures their convergence with the same order of accuracy /10/.

REFERENCES

1. UMANSKII S.E., On the determination of residual stresses occurring during the cooling of viscoelastic solids, Dokl. Akad. Nauk USSR, Ser. A, No.12, 1976.
2. POBEDRIA B.E., Mathematical theory of nonlinear viscoelasticity, IN: Elasticity and Inelasticity, No.3, Izd. Mosk. Univ. 1973.
3. CIOBANU Gh., On the existence and uniqueness of solution in linear viscoelasticity: First boundary value problem. Arch. Mech. Stosowanej, Vol.23, No.1, 45—54, 1971.
4. FICHERA G., Existence theorems in elasticity. IN: Handbuch der Physik, Bd. 6a/2. Springer —Verlag, Berlin, 1972.
5. LADYZHENSKAYA O.A., and URAL'TSEVA N.N., Linear and Quasi-linear Equations of Elliptic Type. "Nauka", Moscow, 1973.
6. DALETSKII, Iu. L. and KREIN M.G., Stability of Solutions of Differential Equations in Banach Space. "Nauka", Moscow, 1970.
7. GAIEWSKI, H., GROGER K. and ZAHARIAS K., Nichtlineare Operatorgleichungen und Operator-Differentialgleichungen. Academic Verlag, Berlin, 1974.
8. ZIENKIEWICZ O., Method of Finite Elements in Engineering, "Mir", Moscow, 1975.
9. STRANG G. and FIX G.J., An Analysis of the Finite Element Method. Prentice—Hall, Englewood Cliffs, N.J., 1973.
10. SAMARSKII A.A., Introduction to the Theory of Difference Schemes. "Nauka", Moscow, 1971.

Translated by M.D.F.